Tutorial 2: Discrete-time Markov chains

In this tutorial, we will see examples of discrete-time Markov chains and do some relevant computations.

Consider some physical system in our concern. Let S be a finite or countably infinite set called state space, whose elements represent possible states of the system, and let X_n be the random variable which represents the state of the system at time n.

Recall that

Definition 1. $\{X_n\}_{n\geq 0}$ *is a* ***Markov chain*** *if it satisfies*

$$\mathbb{P}(X_{n+1} = x_{n+1}|X_0 = x_0, \ldots, X_n = x_n) = \mathbb{P}(X_{n+1} = x_{n+1}|X_n = x_n)$$

i.e. the information needed to predict a future state is wholly summarized in the present state.

and

Definition 2. *A Markov chain* $\{X_n\}_{n\geq 0}$ *is said to be* ***time-homogeneous*** *if the transition probability*

$$\mathbb{P}(X_{n+1} = y|X_n = x)$$

does not depend on n.

The transition matrix is defined by $P = \{p_{ij}\}$ *where*

$$p_{ij} := \mathbb{P}(X_1 = j|X_0 = i), \quad i, j \in S.$$

The initial distribution of the Markov chain is defined by

$$\pi_0(x) := \mathbb{P}(X_0 = x), \quad x \in S.$$

Example 1 (two-state Markov chains)**.** *Consider a basketball player who is now about to make a shot. If he misses the shot, we denote this outcome by* 0*; and if he hits the shot, we denote this outcome by* 1*. Thus, this is a "system" with exactly two possible states.*

Now, take your favourite basketball player e.g. LeBron James and suppose that by our long-time observation, his performance in the court has the following patterns:

1. *If he missed the last shot, then the probability that he will hit the next shot will be* p.
2. *If he hit the last shot, then the probability that he will miss the next shot will be* q.
3. *The probability that he misses his first shot in the game is* $\pi_1(0)$.

(Here we will assume that $p, q \in [0, 1]$ *with* $0 < p + q < 2$*.)*

Mathematically, these information can be summarized as: Let $X_n, n \geq 1$ *denotes the* n*-th shot he makes in a particular game, we have that*

$$\begin{aligned} \mathbb{P}(X_{n+1} = 1|X_n = 0) &= p \\ \mathbb{P}(X_{n+1} = 0|X_n = 1) &= q \\ \mathbb{P}(X_1 = 0) &= \pi_1(0) \end{aligned}$$

This is a Markov chain because the prediction about his next shot depends only on the outcome of his last shot.

Exercise 1. *Suppose that you are listening to the radio for LeBron's game versus Kobe and the player who can first hit three shots will win the game.*

You heard that he misses his first shot, so you become upset and went away to complain it to your friends. Then you come back and hear that he just missed the third shot. Now it is natural for you to wonder if he missed his second shot as well, because you would very much like to know that if he is going to be the loser.

However, you do not have the ability to go back in time and this is why this course is going to be useful for you.

Given the above information, please compute the probability that LeBron missed his second shot.

Solution. This is to find the conditional probability

$$\mathbb{P}(X_2 = 0 | X_1 = 0, X_3 = 0) = \frac{\mathbb{P}(X_1 = 0, X_2 = 0, X_3 = 0)}{\mathbb{P}(X_1 = 0, X_3 = 0)}.$$

The key to evaluate the probabilities is to invoke the Markov property and the trick is to write them in a "Markov" form. Indeed,

$$\begin{aligned}
\mathbb{P}(X_1 = 0, X_2 = 0, X_3 = 0) &= \mathbb{P}(X_1 = 0, X_2 = 0) \cdot \mathbb{P}(X_3 = 0 | X_1 = 0, X_2 = 0) \\
&= \mathbb{P}(X_1 = 0) \cdot \mathbb{P}(X_2 = 0 | X_1 = 0) \cdot \mathbb{P}(X_3 = 0 | X_1 = 0, X_2 = 0) \\
&= \pi_1(0) \cdot [1 - \mathbb{P}(X_2 = 1 | X_1 = 0)] \cdot \mathbb{P}(X_3 = 0 | X_2 = 0) \\
&= \pi_1(0) \cdot [1 - \mathbb{P}(X_2 = 1 | X_1 = 0)] \cdot [1 - \mathbb{P}(X_3 = 1 | X_2 = 0)] \\
&= \pi_1(0) \cdot (1 - p)^2
\end{aligned}$$

and similarly for the denominator we have

$$\begin{aligned}
\mathbb{P}(X_1 = 0, X_3 = 0) &= \mathbb{P}(X_1 = 0, X_2 = 0, X_3 = 0) + \mathbb{P}(X_1 = 0, X_2 = 1, X_3 = 0) \\
&= \pi_1(0) \cdot (1 - p)^2 + \pi_1(0) \cdot pq
\end{aligned}$$

Hence, the probability that LeBron missed his second shot is

$$\mathbb{P}(X_2 = 0 | X_1 = 0, X_3 = 0) = \frac{(1 - p)^2}{(1 - p)^2 + pq}.$$

□

Remark. If you do not like LeBron and wish him to lose, then you may think about for what values of p and q will make him highly likely have missed the second shot (and hence lose the match to Kobe)?

Exercise 2. *Suppose that LeBron just lost to Kobe in game 1. Now that they are going to play against each other again in game 2, and you are making a bet with your friends that this time Lebron will successfully hit the n-th shot.*

Given the above information, how would you predict your winning probability?

Solution. This is to compute the probability row vector

$$\pi_n = \begin{bmatrix} \mathbb{P}(X_n = 0) & \mathbb{P}(X_n = 1) \end{bmatrix}.$$

Recall from the lecture note that

$$\pi_n = \pi_1 P^{n-1}$$

where P is the transition matrix. Since we already know that

$$\pi_1 = \begin{bmatrix}\mathbb{P}(X_1 = 0) & \mathbb{P}(X_1 = 1)\end{bmatrix} = \begin{bmatrix}\pi_1(0) & 1 - \pi_1(1)\end{bmatrix},$$

it remains to compute P^{n-1}.

The transition matrix is just

$$P = \begin{array}{c} \\ \text{Last Shot} \end{array} \begin{array}{c} \\ 0 \\ 1 \end{array} \overset{\text{Next Shot}}{\begin{array}{cc} 0 & 1 \\ \end{array}} \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$$

To find its power, let us diagonalize it:

$$P = QDQ^{-1}$$

where

$$Q = \begin{pmatrix} 1 & -p \\ 1 & q \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 1-p-q \end{pmatrix}.$$

So we have

$$P^n = QD^nQ^{-1} = \begin{pmatrix} \frac{p}{p+q}(1-p-q)^{n-1} + \frac{q}{p+q} & \frac{p}{p+q}(1-(1-p-q)^{n-1}) \\ \frac{q}{p+q}(1-(1-p-q)^{n-1}) & \frac{p}{p+q} + \frac{q}{p+q}(1-p-q)^{n-1} \end{pmatrix}$$

and hence that

$$\pi_n = \pi_1 P^{n-1} = \begin{bmatrix} \frac{q}{p+q} + (\pi_1(0) - \frac{q}{p+q})(1-p-q)^{n-1} & \frac{p}{p+q} - (\pi_1(0) - \frac{q}{p+q})(1-p-q)^{n-1}) \end{bmatrix}$$

□

Example 2 (the Ehrenfest chain)**.** *The Ehrenfest chain is a Markov chain* $\{X_n\}_{n\geq 0}$ *on the state space* $S = \{0, 1, \ldots, d\}$ *with transition probability*

$$P(x,y) = \begin{cases} \frac{x}{d} & \textit{if } y = x-1 \\ \frac{d-x}{d} & \textit{if } y = x+1 \\ 0 & \textit{otherwise} \end{cases}$$

for $x, y \in S$*. (see page 7 of the textbook)*

Exercise 3. *Suppose that* $\mathbb{P}(X_0 = j) = 1$ *for some* $0 < j < d$*. Find* $\mathbb{P}(X_0 = X_2)$*.*

Solution. Indeed, by the Markov property and the definition of $P(x,y)$, we have

$$\begin{aligned}
\mathbb{P}(X_0 = X_2) &= \mathbb{P}(X_0 = j,\ X_2 = j) \\
&= \sum_{k=0}^{d} \mathbb{P}(X_0 = j,\ X_1 = k,\ X_2 = j) \\
&= \sum_{k=0}^{d} \mathbb{P}(X_2 = j \mid X_1 = k,\ X_0 = j) \cdot \mathbb{P}(X_0 = j,\ X_1 = k) \\
&= \sum_{k=0}^{d} \mathbb{P}(X_2 = j \mid X_1 = k) \cdot \mathbb{P}(X_1 = k \mid X_0 = j) \cdot \mathbb{P}(X_0 = j) \\
&= \sum_{k=0}^{d} \mathbb{P}(k,j) \cdot \mathbb{P}(j,k) \cdot 1 \\
&= \mathbb{P}(j-1,j) \cdot \mathbb{P}(j,j-1) + \mathbb{P}(j+1,j) \cdot \mathbb{P}(j+1,j) \\
&= \frac{d-j+1}{d}\frac{l}{d} + \frac{j+1}{d}\frac{d-j}{d} = \frac{(1+2j)d - 2j^2}{d^2}
\end{aligned}$$

□